

Rényi Entropy for the $\widehat{\text{SU}}(N)_1$ WZW model on the torus

Howard J. Schnitzer

*Martin Fisher School of Physics, Brandeis University,
Waltham, MA 02454, USA*

E-mail: schnitzr@brandeis.edu

ABSTRACT:

The $\widehat{\text{SU}}(N)_1$ WZW model is constructed on a n -sheeted branched torus, which allows the investigation of the Rényi entropy for a single interval at finite temperature. The small and large interval limits, as well as the low temperature expansion are presented for this theory.

Contents

1. Introduction	1
2. The Action	3
3. Twist Operators	4
4. The partition function	6
4.1 The classical solution	6
4.2 The quantum solution	10
4.3 Partition function on the n-sheeted torus	10
4.4 Vacuum module	10
5. Some limits	11
5.1 Small interval limit	11
5.2 Large interval limit	12
5.3 Thermal entropy	12
6. Low temperature limit	13
7. Discussion	13
A. W-functions	14

1. Introduction

Entanglement entropy is an important tool in understanding various aspects of gravitational and quantum field theories. The entanglement entropy of a subsystem A is defined by the von Neumann entropy of the reduced density matrix

$$S_{EE} = -\text{Tr} \rho_A \log \rho_A \quad (1.1)$$

At finite temperature (1.1) has been studied for a variety of systems, which is a focus of this paper. In quantum field theory it is convenient to compute the Rényi entropy obtained by means of the replica trick, where the Rényi entropy is

$$S_n = -\frac{1}{n-1} \log \text{Tr} \rho_A^n \quad (1.2)$$

and is related to the entanglement entropy by

$$S_{EE} = \lim_{n \rightarrow 1} S_n \quad (1.3)$$

However such an analytic continuation may not be readily available, which is the case for the $\widehat{\text{SU}}(N)_1$ WZW model. Therefore we are restricted to computing the Rényi entropy.

In two dimensional space-time one can consider the submanifold as a single interval, which we do. The two-dimensional Euclidean field theory is defined on a complex plane, so that the Rényi entropy becomes

$$S_n = -\frac{1}{n-1} \log\left(\frac{Z_n}{Z_1^n}\right) \quad (1.4)$$

where Z_n is the partition function for the n -sheeted Riemann surface which results from connecting the n -complex planes along the branch cuts, i.e. the interval. For a single interval at finite temperature the entanglement and Rényi entropies depend on the details of the CFT. We are particularly interested in $\widehat{\text{SU}}(N)_1$ WZW theory, as it will allow future consideration of two important issues 1) the bose-fermi equivalence for the Rényi entropy and 2) the appropriate topological holographic dual. Both of these are under active consideration.

However $\widehat{\text{SU}}(N)_1$ is interesting in its own right, as distinct from other 2d CFT of free bosons, as the WZW models provide results which present a challenge for holography. Certainly, the usual $\widehat{\text{SU}}(N)_1$ WZW theory is dual to a Chern-Simons theory. Is this true for the Rényi entropy for one or more intervals?

The bose-fermi equivalence of the Rényi entropy for $\widehat{\text{SU}}(N)_1$ raises the issue of the appropriate sum over characters in the fermi presentation, so as to coincide with that of the bose presentation. In this paper we focus on the bose construction of the Rényi entropy for $\widehat{\text{SU}}(N)_1$ and discuss the fermi construction in future work.

There has been considerable work presenting material relevant and useful for this paper [1–29]. The bosonic formulation of $\widehat{\text{SU}}(N)_1$ is based on secs 4 and 5 of Naculich and Schnitzer [30]. See also [21,22]. We are particularly indebted to refs [1–5,9] where we often quote needed results from those papers, without repeating their derivations, in the interest of efficient discussion.

In Sec. 2, we present the bosonic action for $\widehat{\text{SU}}(N)_1$ based on [30]. A number of necessary group theoretic definitions and conventions are to be found in that section. Section 3 presents twist operators appropriate to $\widehat{\text{SU}}(N)_1$. In section 4, we present the classical and quantum solutions for the n -sheeted torus, resulting in the partition function $Z_n = Z_{n,\text{classical}} Z_{n,\text{quantum}}$. Since $Z_{n,\text{classical}}$ depends on the square of Riemann-Siegel theta functions $|\Theta(0|i\Omega)|^2$, continuation to $n \rightarrow 1$ was not possible, so that we are unable to obtain the entanglement entropy. .

In Sec. 5, we present the small interval and large interval limits and in Sec. 6 the low-temperature expansion of the theory. A summary of relevant issues is found in Sec. 7.

2. The Action

We begin with the action for $\widehat{\text{SU}}(N)_1$ and toroidal compactification, following Sec. 5 of ref [30]. Consider a free bosonic field φ^μ valued on a d-dimensional torus $T^d = \mathbb{R}^d/2\pi\Lambda$, with the torus obtained by identifying points in \mathbb{R}^d , differing by a point on the lattice $2\pi\Lambda$, where $\Lambda \subset \mathbb{R}^d$, generated by basis vectors e_i , $i = 1$ to d . The dual vectors e^{*i} are defined by $e_i^\mu e_\mu^{*j} = \delta_i^j$, where the metric on \mathbb{R}^d is $\delta_{\mu\nu}$, $\mu, \nu = 1$ to d . The bosonic field satisfies $\varphi^\mu \sim \varphi^\mu + 2\pi n^i e_i^\mu$, for any set of integers n^i . The action for this field is

$$S[\varphi] = \frac{1}{2\pi} \int (2idz \wedge d\bar{z})(\partial\varphi^\mu \bar{\partial}\varphi^\mu + B_{\mu\nu} \partial\varphi^\mu \bar{\partial}\varphi^\nu) \quad (2.1)$$

where $B_{\mu\nu}$ is a constant anti-symmetric tensor field. Define the metric $g_{ij} = e_i^\mu e_j^\mu$, with inverse $g^{ij} = e_\mu^{*i} e_\mu^{*j}$, so that the action becomes

$$S[\varphi] = \frac{1}{2\pi} \int (2idz \wedge d\bar{z})(g_{ij} + b_{ij}) \partial\varphi^i \bar{\partial}\varphi^j \quad (2.2)$$

where $\varphi^\mu = \varphi^i e_i^\mu$ and $B_{\mu\nu} = b_{ij} e_\mu^{*i} e_\nu^{*j}$

Let Λ_R be the root lattice of $\text{SU}(N)$, with simple roots α_i^μ normalized such that the Cartan matrix is

$$M_{ij} = \alpha_i \cdot \alpha_j = \begin{cases} 2 & \text{for } i = j \\ -1 & \text{for } |i - j| = 1 \\ 0 & \text{for otherwise} \end{cases} \quad (2.3)$$

The weight lattice Λ_W is generated by the weight vectors ω_μ^i dual to α_i^μ such that

$$\begin{aligned} \omega_i \cdot \alpha_j &= \omega_\mu^i \alpha_j^\mu = \delta_{ij} \quad \text{and} \\ \omega_i \cdot \omega_j &= \min(i, j) - \frac{ij}{N} \end{aligned} \quad (2.4)$$

The action (2.1) is equivalent to that of $\widehat{\text{SU}}(N)_1$ if the lattice Λ_W is one-half the root lattice of $\text{SU}(N)$, so that $e_i^\mu = \frac{1}{2}\alpha_i^\mu$, $e_\mu^{*i} = 2\omega_\mu^i$, $g_{ij} = \frac{1}{4}M_{ij}$, and

$$b_{ij} = \begin{cases} \frac{1}{4}M_{ij} & \text{if } i < j \\ 0 & \text{if } i = j \\ -\frac{1}{4}M_{ij} & \text{if } i > j \end{cases} \quad (2.5)$$

The central charge of the theory is $c = N - 1$, and $\text{tr}M = 2c$. Each element of a representation R_I of $\text{SU}(N)$ is associated with a weight vector $\omega_\mu^I \in \Lambda_W$. In fact, the only allowed representations of $\widehat{\text{SU}}(N)_1$ are the N unitary (integrable) representations \hat{R}_r , $r = 0, \dots, N-1$ which transform as the r^{th} fundamental representation of $\text{SU}(N)$ whose Young tableau is a single column of r boxes.

In this paper we consider the single interval Rényi entropy on a two-dimensional torus for $\widehat{\text{SU}}(N)_1$. This torus will allow the consideration of Rényi entropy at finite temperature, and should not be confused with the torus of the target space of φ^μ . One adopts the replica trick to compute $\text{tr}\rho_A^n$ for integer n , where $\text{tr}\rho_A^n$ is the partition function on an n -sheeted Riemann surface obtained by joining successive n -sheets along the region A . For a single interval we denote the surface as $\mathcal{R}_{n,1}$. Thus, we are considering an n -sheeted branched torus.

The Lagrangian density does not depend explicitly on the Riemann surface, so that the structure of $\mathcal{R}_{n,1}$ is implemented by appropriate boundary conditions. The partition function is

$$Z_{\mathcal{R}_{n,1}} = \int [d\varphi] \exp\{-S_n[\varphi]\} \quad (2.6)$$

where $S_n[\varphi]$ is obtained from the Lagrangian density, now evaluated on the Riemann surface $\mathcal{R}_{n,1}$.

Following ref [4,5], we consider n -independent copies of $\widehat{\text{SU}}(N)_1$ and the partition function (2.6) rewritten as a path integral on the n -sheeted torus

$$Z_{\mathcal{R}_{n,1}} = \int_{\mathbb{C}_1} [d\varphi_1 \cdots d\varphi_n] \exp \left\{ -\frac{1}{2\pi} \int_{\mathbb{C}} (2idz \wedge d\bar{z}) [\mathcal{L}[\varphi_1] + \cdots + \mathcal{L}[\varphi_n]] \right\} \quad (2.7)$$

where $\int_{\mathbb{C}_1}$ is the restricted path integral with boundary conditions

$$\varphi_i(x, 0^+) = \varphi_{i+1}(x, 0^-), \quad (2.8)$$

where one identifies $n+1 = 1$.

3. Twist Operators

Twist fields enforce two opposite permutation symmetries

$$i \rightarrow i+1, \quad \text{and} \quad i+1 \rightarrow i \quad (i = 1, \cdots, n),$$

identifying $n+1 \equiv 1$. Thus

$$\begin{aligned} T_n &= i \rightarrow i+1 \pmod{n} \\ \tilde{T}_n &= i+1 \rightarrow i \pmod{n} \end{aligned} \quad (3.1)$$

where \tilde{T}_n can be identified with T_{-n} .

Thus for the j^{th} sheet, circling the branch point (z, \bar{z}) , one has

$$\varphi_j^\mu(z e^{2\pi i}, \bar{z} e^{-2\pi i}) = \varphi_{j+1}^\mu(z, \bar{z}) \quad (3.2)$$

which is implemented by T_n .

It is useful to introduce the linear combination

$$\tilde{\varphi}_k^\mu = \sum_{j=0}^{n-1} (e^{2\pi i k/n})^j \varphi_j^\mu \quad (3.3)$$

which gets multiplied by $e^{2\pi i k/n}$ circling the twist operator. Thus (3.3), diagonalizes the twist

$$\begin{aligned} T_n \tilde{\varphi}_k^\mu &= e^{2\pi i k/n} \tilde{\varphi}_k^\mu \\ \tilde{T}_n \tilde{\varphi}_k^\mu &= e^{-2\pi i k/n} \tilde{\varphi}_k^\mu \end{aligned} \quad (3.4)$$

One can write

$$\begin{aligned} T_n &= \prod_{k=0}^{n-1} T_{n,k} \quad \text{and} \\ \tilde{T}_n &= \prod_{k=0}^{n-1} \tilde{T}_{n,k} \end{aligned} \quad (3.5)$$

with the action of $T_{n,k}$ diagonal. For a single scalar field

$$\begin{aligned} T_{n,k} \tilde{\varphi}_{k'} &= \tilde{\varphi}_{k'} \quad \text{if} \quad k \neq k' \quad \text{and} \\ T_{n,k} \tilde{\varphi}_k &= e^{2\pi i k/n} \tilde{\varphi}_k \end{aligned} \quad (3.6)$$

Define

$$\begin{aligned} \theta_k &= e^{2\pi i k/n}, \quad \text{so that} \\ \sum_{j=0}^{n-1} (\theta_k)^j &= 0 \end{aligned} \quad (3.7)$$

For $\widehat{\text{SU}}(N)_1$ the scalar fields are valued on the lattice Λ_W , so that one can write

$$\tilde{\varphi}_k^\mu(z, \bar{z}) = \tilde{\varphi}_k^a(z, \bar{z}) e_a^\mu \quad a = 1 \text{ to } N-1 \quad (3.8)$$

It is useful to further diagonalize the twist operator with respect to the “color”. That is

$$T_{n,k} = \prod_{i=1}^{N-1} T_{n,k}^{(i)} \quad (3.9)$$

where

$$T_{n,k}^{(i)} = \exp[i\zeta_\mu^{(i)}(k) \tilde{\varphi}_k^\mu(z)] \quad (3.10)$$

with

$$\zeta_\mu^{(i)}(k) = \frac{1}{2}(k/n)\alpha_\mu^i; \quad \mu = 1 \text{ to } N-1 \quad (3.11)$$

The two-point function satisfies

$$\lim_{z \rightarrow z'} \langle \tilde{\varphi}_k^\mu(z) \tilde{\varphi}_l^\nu(z') \rangle = -\delta_{kl} g^{\mu\nu} \ln |z - z'| \quad 0 \leq k, l \leq n-1 \quad (3.12)$$

Then the quantum dimension of $T_{n,k}$ is obtained from

$$\begin{aligned} \lim_{z \rightarrow z'} \prod_{i=1}^{N-1} \langle T_{n,k}^{(i)}(z) \tilde{T}_{n,k}^{(i)}(z') \rangle &= \exp - \left[\sum_{i=1}^{N-1} \frac{1}{4} (k/n)^2 M_{ii} \ln |z - z'| \right] \\ &= |z - z'|^{-\Delta_{k/n}} \end{aligned} \quad (3.13)$$

$$\text{with} \quad \Delta_{k/n} = \frac{c}{2} (k/n)^2 \quad (3.14)$$

where $\text{tr} M = 2c$ has been used, and $c = N - 1$.

The scalar fields can be divided into a classical and quantum part, $\hat{\phi}_k^\mu$ and ϕ_k^μ respectively, where

$$\tilde{\varphi}_k^\mu(z, \bar{z}) = \hat{\phi}_k^\mu(z, \bar{z}) + \phi_k^\mu(z, \bar{z}) \quad (3.15)$$

where the quantum part is independent of the lattice contribution in computing the monodromies of the fields around a branch point. We examine these two contributions separately in the next section.

4. The partition function

4.1 The classical solution

From (3.15), the classical solution satisfies

$$\phi_k^\mu(z e^{2\pi i}, \bar{z} e^{-2\pi i}) = \theta_k \phi_k^\mu(z, \bar{z}) + v_{(k)}^\mu \quad (4.1)$$

where $v_{(k)}^\mu \in \Lambda_{k/n}^\mu$, with the lattice defined by

$$\Lambda_{k/n}^\mu = \left\{ q^\mu = \sum_{j=0}^{n-1} \theta_k^j n_j^\mu \right\} \quad (4.2)$$

where $\mu = 1$ to $N - 1$ and $n_j^\mu \in \mathbb{Z}$. This generalizes eqns. (30) and (31) of [4]. The monodromy conditions satisfied by the classical solution on the world sheet are

$$\oint_{\gamma_a} dz \partial \phi_k^\mu + \oint_{\gamma_a} d\bar{z} \bar{\partial} \phi_k^\mu = v_{(k),a}^\mu \quad (4.3)$$

$$\oint_{\gamma_a} dz \partial \bar{\phi}_k^\mu + \oint_{\gamma_a} d\bar{z} \bar{\partial} \bar{\phi}_k^\mu = \bar{v}_{(k),a}^\mu \quad (4.4)$$

where $\bar{\phi}_k^\mu$ is the complex conjugate of ϕ_k^μ , and γ_a are the two cycles of the world-sheet torus with $a = 1, 2$.

The classical solution can be expressed in terms of the cut-differentials [see Appendix A] [1, 2]

$$\begin{aligned} \partial\phi_k^\mu &= a_{(k)}^\mu \omega_1^{(k)}(z) \\ \bar{\partial}\phi_k^\mu &= b_{(k)}^\mu \bar{\omega}_2^{(k)}(\bar{z}) \\ \partial\bar{\phi}_k^\mu &= \bar{a}_{(k)}^\mu \omega_2^{(k)}(z) \\ \bar{\partial}\bar{\phi}_k^\mu &= \bar{b}_{(k)}^\mu \bar{\omega}_1^{(k)}(\bar{z}) \end{aligned} \quad \mu = 1 \quad \text{to} \quad N-1 \quad (4.5)$$

The solution to monodromy conditions are [1–3, 9]

$$\begin{aligned} a_{(k)}^\mu &= [W_2^{2(k)} \bar{v}_{(k),1}^\mu - W_1^{2(k)} v_{(k),2}^\mu] / \det W^{(k)} \\ \bar{a}_{(k)}^\mu &= [\bar{W}_1^{1(k)} \bar{v}_{(k),2}^\mu - \bar{W}_1^{1(k)} \bar{v}_{(k),1}^\mu] / \det \bar{W}^{(k)} \\ b_{(k)}^\mu &= [-W_2^{1(k)} v_{(k),1}^\mu + W_1^{1(k)} v_{(k),2}^\mu] / \det W^{(k)} \\ \bar{b}_{(k)}^\mu &= [\bar{W}_2^{2(k)} \bar{v}_{(k),1}^\mu - \bar{W}_1^{2(k)} \bar{v}_{(k),2}^\mu] / \det \bar{W}^{(k)} \end{aligned} \quad (4.6)$$

Next we present the Lagrange density and classical action appropriate to (4.5), and (4.6). The Lagrange density (2.7) for the classical solution leads to the classical portion of the action at fixed k ,

$$S^{(k)} = \frac{4}{n\pi} \frac{W_1^{1(k)} W_2^{2(k)}}{[\det W^{(k)}][\det \bar{W}^{(k)}]} \sum_{a,b=1}^{N-1} (g_{ab} + b_{ab}) [(v_{(k),1}^a \bar{v}_{(k),1}^b) |W_2^{2(k)}|^2 + (v_{(k),2}^a \bar{v}_{(k),2}^b) |W_1^{1(k)}|^2] \quad (4.7)$$

As emphasized by [1, 2], once one fixes the monodromy around the canonical cycles, one can obtain all the monodromies on the Riemann surface. For the n -sheeted torus these can be fixed as

$$\begin{aligned} \oint_{\gamma_1} dz \partial\phi_j^\mu &= 2\pi m_j^\mu \quad \text{and} \\ \oint_{\gamma_2} dz \partial\phi_j^\mu &= 2\pi l_j^\mu \end{aligned} \quad (4.8)$$

where m_j^μ and $l_j^\mu \in \mathbb{Z}$. In the basis in which the twist operators acts as in (3.4) one has

$$\begin{aligned} v_{(k),1}^a &= 2\pi \sum_{j=0}^{n-1} (e^{2\pi i k/n})^j m_j^a \quad \text{and} \\ v_{(k),2}^a &= 2\pi \sum_{j=0}^{n-1} (e^{2\pi i k/n})^j n_j^a \end{aligned} \quad (4.9)$$

The action becomes

$$S^{(k)} = \frac{16\pi}{n} \sum_{a,b=1}^{N-1} (g_{ab} + b_{ab}) \sum_{j=0}^{n-1} \sum_{j'=0}^{n-1} \{ \beta_{k/n} (e^{2\pi i k/n})^{j-j'} (m_j^a m_{j'}^b) + (\beta_{k/n})^{-1} (e^{2\pi i k/n})^{j-j'} (n_j^a n_{j'}^b) \} \quad (4.10)$$

where we have defined

$$\beta_{k/n} = \left| \frac{W_2^{2(k)}}{W_1^{1(k)}} \right| \quad (4.11)$$

Also define

$$\begin{aligned} (C_{k/n})_{jj'} &= \cos\left[\frac{2\pi k}{n}(j - j')\right] \\ (S_{k/n})_{jj'} &= \sin\left[\frac{2\pi k}{n}(j - j')\right] \end{aligned} \quad (4.12)$$

The classical partition function is

$$Z_n^{cl} = \exp - \sum_{k=0}^{n-1} S^k \quad (4.13)$$

The sum over k in the total partition function at fixed (j, j') involves two terms, $S_{k/n}$ and $(S)_{(n-k)/n}$, which cancel in the sum over k . Therefore effectively

$$\begin{aligned} S^{(k)} &= \frac{16\pi}{n} \sum_{j=0}^{n-1} \sum_{j'=0}^{n-1} \sum_{a=1}^{N-1} \{ \beta_{k/n} m_j^a (C_{k/n})_{jj'} m_{j'}^a \\ &\quad + (\beta_{k/n})^{-1} n_j^a (C_{k/n})_{jj'} n_{j'}^a \} \end{aligned} \quad (4.14)$$

Carry out a Poisson re-summation of the second term in (4.14), changing $n_j^a \rightarrow l_j^a$. Then

$$\begin{aligned} Z_n^{(cl)} &= \prod_{k=0}^{n-1} (\beta_{k/n})^{1/2} \sum_{m_j^a, l_j^a} \exp - \frac{16\pi}{n} \sum_{a=1}^{N-1} \sum_{j=0}^{n-1} \sum_{j'=0}^{n-1} (\beta_{k/n}) \{ \vec{m}^a \cdot (C_{k/n}) \cdot \vec{m}^a \\ &\quad + \vec{l}^a \cdot (C_{k/n}) \cdot \vec{l}^a \} \end{aligned} \quad (4.15)$$

where

$$\vec{m}^a \cdot (C_{k/n}) \cdot \vec{m}^a = m_j^a [g_{ab} (C_{k/n})_{jj'}] m_{j'}^b \quad (4.16)$$

so that

$$\vec{m} \in (\mathbb{Z}^n)^{N-1} \quad (4.17)$$

The Riemann-Siegel theta function is

$$\Theta(0|\Gamma) = \sum_{m \in \mathbb{Z}^n} \exp[i\pi m^\dagger \cdot \Gamma \cdot m] \quad (4.18)$$

In our case, from (4.15)

$$\Gamma_{ab,jj'} = \frac{16i}{n} \delta_{ab} \sum_{k=0}^{n-1} \beta_{k/n} (C_{k/n})_{jj'} = i\Omega_{ab,jj'} \quad (4.19)$$

Therefore, the classical partition function is

$$Z_n^{cl} = \left[\prod_{k=0}^{n-1} \beta_{k/n} \right]^{1/2} [\Theta(0|i\Omega)]^2 \quad (4.20)$$

Recall (4.18) and (4.19), and define

$$(p_L^\mu)_s = -m_s^a e_a^\mu \quad (4.21)$$

where μ and $a = 1$ to $N - 1$, and $s = 0$ to $n - 1$. Then write

$$(p_L^\mu)_s = p_s^\mu + \omega_s^\mu \quad (4.22)$$

$$\text{where } p_s^\mu \in (\Lambda_R)^g \quad (4.23)$$

$$\text{and } \omega_s^\mu \in (\Lambda_W^g / \Lambda_R^g) \quad (4.24)$$

where Λ_R^g and Λ_W^g are the root and weight lattices at fixed s , and $g = N - 1$. [Recall Sec. 2] Similarly define

$$(p_R^\mu)_s = l_s^a e_a^\mu \quad (4.25)$$

as well as the analogues of (4.23) and (4.24). Then

$$Z_n^{cl} = \left[\prod_{k=0}^{n-1} \beta_{k/n} \right]^{1/2} \sum_{\vec{\omega}_r} \left| \sum_{\vec{p}_R \in \Lambda_R^g} \exp -\pi \sum_{\mu, \nu=1}^{N-1} \sum_{r,s=0}^{n-1} (p_r + \omega_r)^\mu \Omega_{\mu\nu,r,s} (p_s + \omega_s)^\nu \right|^2 \quad (4.26)$$

where p_s^μ is as in (4.22), (4.23) and ω_r^μ is as in (4.24). [When $n = 1$, this sum goes over holomorphic blocks. See [30], eqn (4.37)]. In particular in (4.26)

$$\vec{\omega}_r = (\omega_{r_1}, \dots, (\omega_r)_{N-1}) \quad r = 0 \text{ to } n - 1 \quad (4.27)$$

where each ω_{r_i} is a fundamental weight corresponding to a $SU(N)$ Young tableau with a single column of boxes.

4.2 The quantum solution

The quantum portion of (3.15) satisfies

$$\hat{\phi}_k^\mu(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = \theta_k \hat{\phi}_k^\mu(z, \bar{z}) \quad (4.28)$$

so that for any closed loop C .

$$\Delta_c \hat{\phi}_k^\mu = \oint_c dz \partial \hat{\phi}_k^\mu + \oint_c d\bar{z} \bar{\partial} \hat{\phi}_k^\mu = 0 \quad (4.29)$$

Following refs [1–5, 9], the quantum contribution to the partition function is

$$Z_{n,k}^{(qu)} = \frac{1}{|\eta(\tau)|^{2(N-1)n}} \prod_{k=0}^{n-1} \left\{ \frac{1}{|W_1^{1(k)} W_2^{2(k)}|^{1/2}} \left[\frac{\theta'_1(0|\tau)}{\theta_1(z_1 - z_2|\tau)} \right]^{\Delta_{k/n}} \left[\frac{\overline{\theta'_1(0|\tau)}}{\overline{\theta_1(z_1 - z_2|\tau)}} \right]^{\Delta_{k/n}} \right\} \quad (4.30)$$

The short distance limit is

$$\left[\frac{\theta'_1(0|\tau)}{\theta_1(l|\tau)} \right]^{\Delta_{k/n}} \longrightarrow l^{-\Delta_{k/n}} \quad \text{as } l \rightarrow 0 \quad (4.31)$$

which is consistent with (3.13)

4.3 Partition function on the n-sheeted torus

Combining equation (4.20) and (4.30), the partition function on the n -sheeted torus is

$$\begin{aligned} Z_n &= Z_n^{(cl)} Z_n^{(qu)} \\ &= \frac{c_n}{|\eta(\tau)|^{2(N-1)n}} \prod_{k=0}^{n-1} \left\{ \frac{1}{|W_1^{1(k)}|} \left| \frac{\theta'_1(0|\tau)}{\theta_1(z_1 - z_2|\tau)} \right|^{2\Delta_{k/n}} |\Theta(0|i\Omega)|^2 \right\} \end{aligned} \quad (4.32)$$

with c_n an overall normalization independent of τ and $(z_1 - z_2)$. This is the main result of this paper.

In the next section we isolate the vacuum module from (4.32), which may be useful in examining holographic duals. Then in Secs. 5 and 6 we consider the small interval, large interval, and low-temperature limits of (4.32), following the strategy of [1, 2].

4.4 Vacuum module

It is interesting to isolate the vacuum module from (4.32), obtained from the identity representation $\omega_r^\mu = 0$ for all r . One can express

$$(p^\mu)_s = \sum_{a=1}^{N-1} (n_a)_s \alpha_a^\mu \quad s = 0 \text{ to } n-1 \quad (4.33)$$

with $(n_a)_s \in \mathbb{Z}$. Then, using the properties of the Cartan matrix

$$(p^\mu)_s (C_{k/n})_{s,t} (p^\mu)_t = 2 \sum_{s,t=0}^{n-1} \left\{ \sum_{a=1}^{N-1} (n_a)_s (n_a)_t - \sum_{a=1}^{N-2} (n_{a+1})_s (n_a)_t \right\} (C_{k/n})_{st} \quad (4.34)$$

One can then consider (4.32) with (4.34) in the large N limit to investigate a holomorphic interpretation of this sub-module [31]. This issue is outside the scope of this paper.

5. Some limits

5.1 Small interval limit

We apply the strategy and equation (A-21) of [1, 2], which gives the small interval limits

$$W_1^{1(k)} = 1 + \mathcal{O}(z_1 - z_2) \quad (5.1)$$

$$W_2^{2(k)} = i\beta + \mathcal{O}(z_1 - z_2) \quad (5.2)$$

Therefore

$$\beta_{k/n} \rightarrow \beta \quad (5.3)$$

in this limit.

Consider the partition function (4.32) on a rectangular torus of spatial size L and Euclidean time β . The Rényi entropy (1.2) is associated with a spatial region A running from 0 to l . From (4.15) to (4.32) with $c = N - 1$,

$$Z_n = \frac{1}{|\eta(\tau)|^{2(N-1)n}} (l/L)^{-\frac{c}{12}n(1-1/n^2)} \sum_{m_j, l_j \in (\mathbb{Z}^{N-1})^n} \exp -\pi [m \cdot \Omega \cdot m + l \cdot \Omega \cdot l] \quad (5.4)$$

Therefore¹

$$\Omega_{ab,jj'} \rightarrow \frac{16 g_{ab} \beta}{n} \sum_{k=0}^{n-1} (C_{k/n})_{jj'} \quad (5.5)$$

Further

$$\sum_{k=0}^{n-1} (e^{2\pi i k/n})^{(j-j')} = n \delta_{jj'}, \quad (5.6)$$

¹Don't confuse the small interval length l with the vector \vec{l} in (5.4)

so that after summing on j and j' ,

$$Z_n \rightarrow \frac{1}{|\eta(\tau)|^{2(N-1)n}} (l/L)^{-\frac{c}{12}n(1-1/n^2)} \left\{ \sum_{m_a, \bar{l}_a \in \mathbb{Z}^{N-1}} \exp -16\pi\beta [\bar{m} \cdot \bar{m} + \bar{l} \cdot \bar{l}] \right\}^n \quad (5.7)$$

Therefore

$$Z_n \rightarrow (l/L)^{-\frac{c}{12}n(1-1/n^2)} [(Z_1)^n + \mathcal{O}(l/L)] \quad (5.8)$$

in agreement with [1–3] for other CFT's. The entanglement entropy can be computed for this limit using (1.4), since one can now carry out the analytic continuation in n .

5.2 Large interval limit

One makes use of various modular transformations

$$\theta_1(-k/n|i\beta) = i\beta^{-1/2} e^{-\frac{k^2}{n^2\beta}} \theta_1\left(\frac{ik/n}{\beta} \middle| \frac{i}{\beta}\right) \quad (5.9)$$

$$\eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau) \quad (5.10)$$

$$\frac{\theta_1'(0|-1/\tau)}{\theta_1(z/\tau|-1/\tau)} = i\tau e^{-i\pi z^2/\tau} \frac{\theta_1'(0|\tau)}{\theta_1(z|\tau)} \quad (5.11)$$

and

$$\left| \frac{W_2^{2(k)}}{W_1^{1(k)}} \right| \rightarrow \left| \frac{W_1^{1(k)}}{W_2^{2(k)}} \right| \quad (5.12)$$

See refs [1, 3, 9]. Then following closely the arguments of [1, 3] applied to (4.32), and the torus of Sec. 5.1, one obtains

$$Z_n \rightarrow (l/L)^{-\frac{c}{12}n(1-1/n^2)} [Z_1(n\beta) + \dots] \quad (5.13)$$

Once again note that the entanglement entropy can be computed in this limit from (1.4), as the n -dependence can be continued to $n \rightarrow 1$.

5.3 Thermal entropy

A straightforward calculation gives

$$\begin{aligned} \lim_{l/L \rightarrow 0} [S_{EE}(1 - l/L) - S_{EE}(l/L)] &= -\lim_{n \rightarrow 1} \frac{1}{n-1} [\log Z_1(n\beta) - n \log Z_1(\beta)] \\ &= \log Z_1(\beta) - \frac{1}{\beta} \frac{Z_1'(\beta)}{Z_1(\beta)} \\ &= S_{th} \end{aligned} \quad (5.14)$$

where S_{th} is the thermal entropy, which is in agreement with other CFT's [1, 3, 6], as well as the holographic discussion [10].

6. Low temperature limit

We begin with (4.18), (4.19) and (4.32) and expand with respect to $q = e^{-2\pi\beta}$, so that

$$\eta(\tau) = q^{1/24}(1 + \mathcal{O}(q)) \quad (6.1)$$

$$\theta'(0) = 2\pi q^{1/8}(1 + \mathcal{O}(q)) \quad (6.2)$$

$$\theta_1(z_1 - z_2|\tau) = 2q^{1/8} \sin \pi(z_1 - z_2)(1 + \mathcal{O}(q)) \quad (6.3)$$

From ref [1]², their equations (3.5) and (A.3),

$$W_1^{1(k)} \rightarrow 1 + \mathcal{O}(q), \quad \text{and} \quad (6.4)$$

$$\frac{W_2^{2(k)}}{W_1^{1(k)}} = i\beta + \int_{z_1}^{z_2} dt \frac{2i[\sin \pi(k/n)(t - z_1)][\sin \pi(1 - k/n)(t - z_1)]}{\sin \pi(t - z_1)} + \mathcal{O}(q) \quad (6.5)$$

The leading and next to leading contribution to (4.18) and (4.19) come from all $m_a = 0$ and $m_a = \pm 1$ respectively, with $2n$ choices for the later. Therefore

$$\Theta(0|i\Omega) \approx 1 + 2n(N - 1)e^{-\pi\beta} \left[\frac{\sin \pi(z_2 - z_1)}{n \sin \frac{\pi}{n}(z_2 - z_1)} \right] + \mathcal{O}(e^{-2\pi\beta}) \quad (6.6)$$

Using this in conjunction with (4.32)

$$Z_n \sim c_n \frac{1}{q^{n/6}} \left(\frac{\pi}{\sin \pi l/L} \right)^{\frac{c}{12}n(1-1/n^2)} \left\{ 1 + 4n(N - 1)e^{-\pi\beta} \left[\frac{\sin \pi(l/L)}{n \sin \frac{\pi}{n}(l/L)} \right] \right\} \quad (6.7)$$

where c_n is an overall normalization constant which does not depend on q or $(z_1 - z_2)$. Therefore, the Rényi entropy in this limit is

$$S_n = \tilde{c}_n - \frac{1}{n-1} 4n(N - 1) \left(\left[\frac{\sin \pi(l/L)}{n \sin \frac{\pi}{n}(l/L)} \right] - 1 \right) e^{-\pi\beta} + \dots \quad (6.8)$$

where \tilde{c}_n is an overall normalization, where (6.8) is consistent with the universal behavior predicted by Cardy and Herzog [7].

7. Discussion

In this paper we considered the $\widehat{\text{SU}}(N)_1$ WZW model on a n -sheeted branched torus, which allows the computation of the Rényi entropy for a single interval at finite temperature for the 2d CFT. The n -sheeted partition function Z_n is given in terms of Riemann-Siegel functions, which then does not allow an obvious analytic continuation

²The discussion of this section follows closely that of [1], section 3.

for $n \rightarrow 1$, which thus prevents a computation of the entanglement entropy. However such an analytic continuation is possible for small and large interval limits, and the leading term of the low-temperature expansion.

The discussion of the $\widehat{\text{SU}}(N)_1$ WZW model in this paper was given entirely in the bosonic formulation. It is known that there is a bose-fermi equivalence for the $\widehat{\text{SU}}(N)_1$ WZW model for Z_1 on a genus g surface [30]. The fermi presentation of $\widehat{\text{SU}}(N)_1$ allows for discussion of the Rényi entropy in the presence of a continuum constraint gauge field. This is presently under investigation.

It is known that for $n = 1$, $\widehat{\text{SU}}(N)_1$ is dual to a Chern-Simons topological theory. The same issue arises for the Rényi entropy of the theory and its dual interpretation. What is the detailed Chern-Simons dual relevant to the construction of this paper? See [32] for a discussion of a related issue.³ In Sec. 4D we isolated the Rényi entropy of the vacuum subspace of $\widehat{\text{SU}}(N)_1$. Does this by itself have a holographic interpretation?

Another application of WZW models to problems of entanglement is [33] where left-right entanglement and level-rank duality is discussed. Bose-fermi duality and entanglement entropies for a massless Dirac fermion and a compact free boson in two dimensions is considered in [18].

A. W-functions

The contents of this Appendix are from equation (2.9), (A.1) and (A.2) of [1],

Cut differentials are defined by (2.9) of [1]

$$\begin{aligned}\omega_1^{(k)}(z) &= \theta_1(z - z_1|\tau)^{-(1-\frac{k}{n})} \theta_1(z - z_2|\tau)^{-\frac{k}{n}} \theta_1(z - (1 - \frac{k}{n})z_1 - \frac{k}{n}z_2|\tau) \\ \omega_2^{(k)}(z) &= \theta_1(z - z_1|\tau)^{-\frac{k}{n}} \theta_1(z - z_2|\tau)^{-(1-\frac{k}{n})} \theta_1(z - \frac{k}{n}z_1 - (1 - \frac{k}{n})z_2|\tau)\end{aligned}\quad (\text{A.1})$$

The W-functions are given by equations (A.1) and (A.2) of [1]

$$\begin{aligned}W_1^{1(k)} &= \int_0^1 dz \theta_1(z - z_1|\tau)^{-(1-\frac{k}{n})} \theta_1(z - z_2|\tau)^{-\frac{k}{n}} \theta_1(z - (1 - \frac{k}{n})z_1 - \frac{k}{n}z_2|\tau) \\ W_1^{2(k)} &= \int_0^1 d\bar{z} \bar{\theta}_1(\bar{z} - \bar{z}_1|\tau)^{-\frac{k}{n}} \bar{\theta}_1(\bar{z} - \bar{z}_2|\tau)^{-(1-\frac{k}{n})} \bar{\theta}_1(\bar{z} - \frac{k}{n}\bar{z}_1 - (1 - \frac{k}{n})\bar{z}_2|\tau) \\ W_2^{1(k)} &= \int_0^\tau dz \theta_1(z - z_1|\tau)^{-(1-\frac{k}{n})} \theta_1(z - z_2|\tau)^{-\frac{k}{n}} \theta_1(z - (1 - \frac{k}{n})z_1 - \frac{k}{n}z_2|\tau) \\ W_2^{2(k)} &= \int_0^{\bar{\tau}} d\bar{z} \bar{\theta}_1(\bar{z} - \bar{z}_1|\tau)^{-\frac{k}{n}} \bar{\theta}_1(\bar{z} - \bar{z}_2|\tau)^{-(1-\frac{k}{n})} \bar{\theta}_1(\bar{z} - \frac{k}{n}\bar{z}_1 - (1 - \frac{k}{n})\bar{z}_2|\tau)\end{aligned}\quad (\text{A.2})$$

and

$$W_1^{1*} = W_1^1 = W_1^2, \quad W_2^{1*} = -W_2^1 = W_2^2. \quad (\text{A.3})$$

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